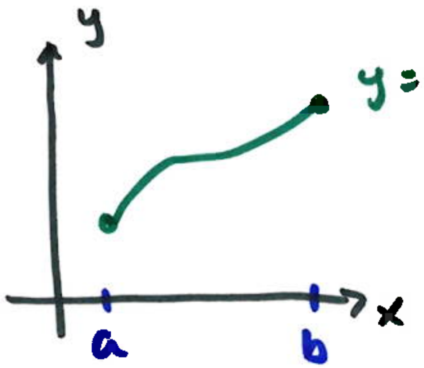
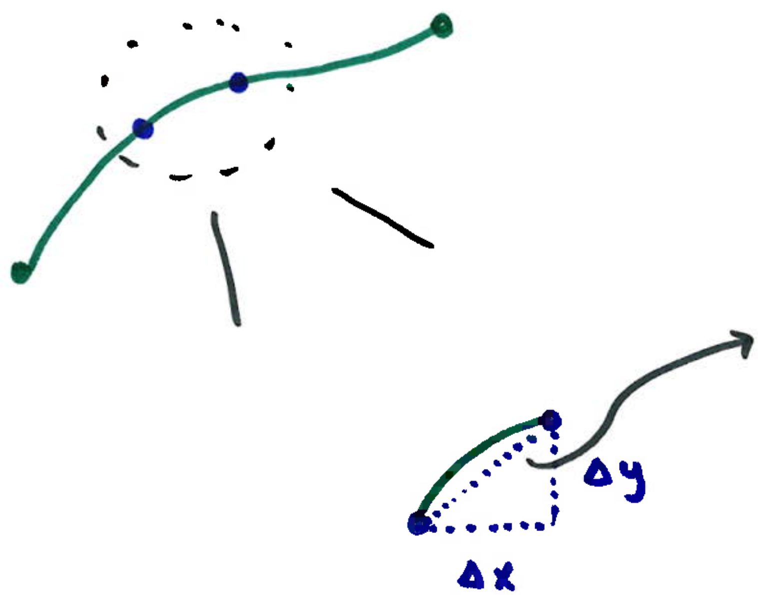
6.5 + 6.6 Length and Surface Area



How long is the length of $f(x)$ from $x = a$ to $x = b$

Same idea as last time: find the length of one small piece then accumulate using integration



is approximately the length of the green piece

the length L of the curve (green)

$$L \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 \left[1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right]} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,(\Delta x)$$

now shrink the interval : $\Delta x \to dx$

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$

so, the exact length of the small piece is $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

now accumulate ALL from $x = a$ to $x = b$

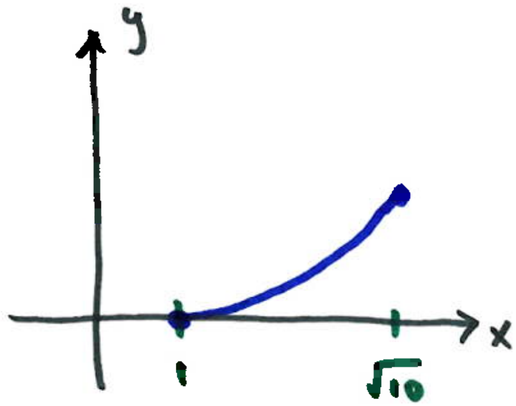
so, the exact length of $y = f(x)$ from $x = a$ to $x = b$ is

$$\boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \quad \text{or} \quad \int_a^b \sqrt{1 + (y')^2}\, dx$$

$y = f(x)$ must be continuous and differentiable on $a \le x \le b$

$f'(x)$ must exist on $a \le x \le b$

example $y = \frac{2}{3}(x^2-1)^{3/2}$ from $x=1$ to $x=\sqrt{10}$



$$L = \int_a^b \sqrt{1+(y')^2}\,dx$$

$$y' = \frac{2}{3}\cdot\frac{3}{2}(x^2-1)^{1/2}\cdot 2x = 2x(x^2-1)^{1/2}$$

$$\int_1^{\sqrt{10}} \sqrt{1+\left[2x(x^2-1)^{1/2}\right]^2}\,dx = \int_1^{\sqrt{10}} \sqrt{1+4x^2(x^2-1)}\,dx$$

$$= \int_1^{\sqrt{10}} \sqrt{\underbrace{4x^4-4x^2+1}_{(2x^2-1)^2}}\,dx = \int_1^{\sqrt{10}} (2x^2-1)\,dx = \cdots = \boxed{\frac{17}{3}\sqrt{10}+\frac{1}{3}}$$

back to $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

now factor out $(\Delta y)^2$

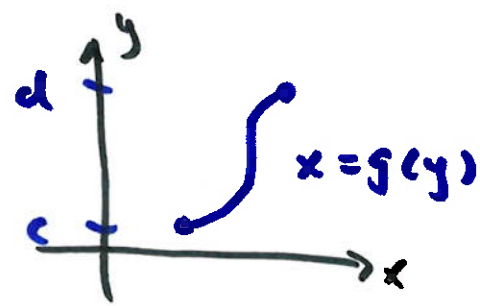
$$\sqrt{(\Delta y)^2 \left(\frac{(\Delta x)^2}{(\Delta y)^2} + 1 \right)}$$

$$= \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2} \, (\Delta y)$$

shrink interval: $\Delta y \to dy$, $\frac{\Delta x}{\Delta y} \to \frac{dx}{dy}$

the equivalent formula for length of $x = g(y)$ from $y = c$ to $y = d$

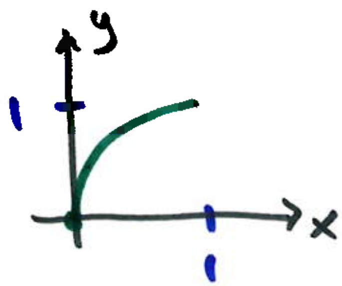
is $$\boxed{\int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy}$$



Sometimes the switch in variable is for convenience

Sometimes we have to switch, for example, when $f'(x)$ does not exist somewhere on $a \le x \le b$ in $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$

<u>example</u> $y = x^{2/3}$ from $x=0$ to $x=1$



$$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

$$\int_0^1 \sqrt{1+\left(\boxed{\frac{2}{3x^{1/3}}}\right)^2}\,dx$$

DNE at $x=0$

try switching to $x = g(y)$

$$y = x^{2/3}$$

$$x = y^{3/2}$$

now use $\displaystyle\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ $\quad c=0,\ d=1$ for this example

(y of starting point is 0
y of ending point is 1)

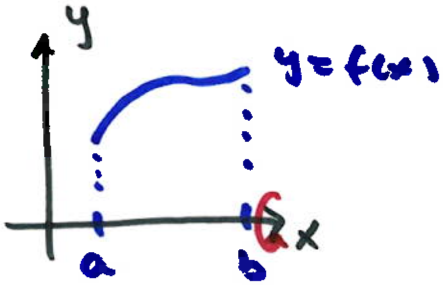
$x = y^{3/2}$ $\quad \dfrac{dx}{dy} = \dfrac{3}{2}y^{1/2}$ which always exists on $0 \le y \le 1$

$$\int_0^1 \sqrt{1+\left(\frac{3}{2}y^{1/2}\right)^2}\,dy = \int_0^1 \sqrt{1+\frac{9}{4}y}\,dy$$

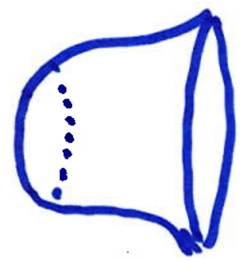
$u = 1+\frac{9}{4}y$ and so on

$$= \cdots = \frac{1}{27}\left(13\sqrt{13}-8\right) \approx 1.44$$

6.6 Surface Area of Solid of Revolution

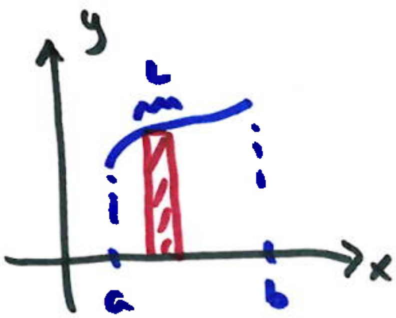


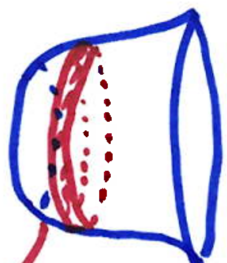
revolve around x-axis



Volume: disk or shell methods

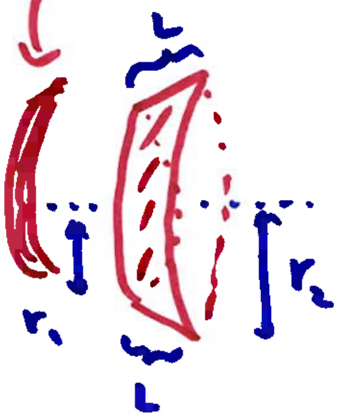
Surface area = ?





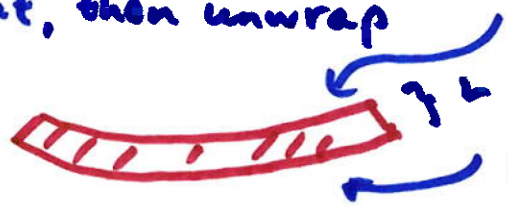
rectangle → strip on surface

idea: find area of one strip
then integrate to accumulate
ALL strips



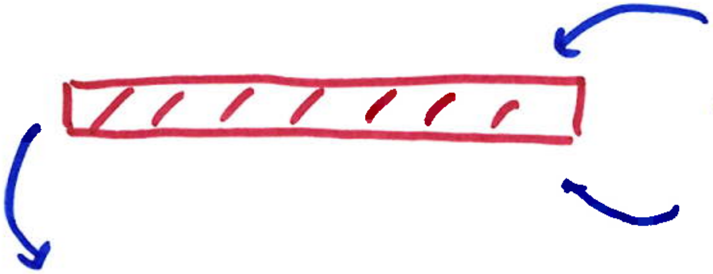
cut, then unwrap

d_1 (diameter from r_1)



d_2 (diameter from r_2)

Shrink interval, the curved strip will be approximately a rectangle



same diameter from $r = f(x)$

$\rightarrow 2\pi \cdot f(x)$

L = length from last section

$$= \sqrt{1 + [f'(x)]^2}\, dx$$

each strip as area $\quad 2\pi \cdot f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$

accumulate from $x = a$ to $x = b$

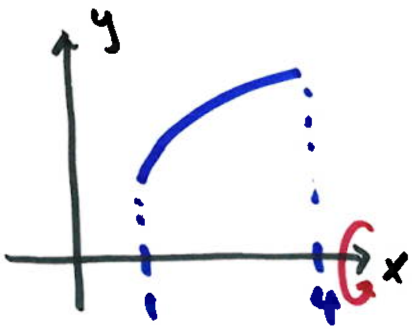
$$\boxed{\int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx}$$

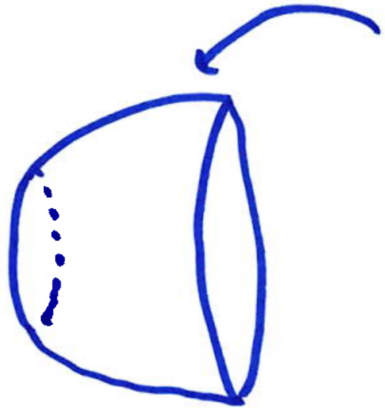
around x-axis

equivalent form
(around y-axis)

$$\int_c^d 2\pi\, g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

example: Region bounded by $y=\sqrt{x}$, $x=1$, $x=4$, $y=0$ revolved around x-axis





surface area

$$\int_a^b 2\pi f(x)\sqrt{1+(f')^2}\,dx$$

$f=\sqrt{x}=x^{1/2}$ $f'=\frac{1}{2}x^{-1/2}$

$=\frac{1}{2\sqrt{x}}$

$$\int_1^4 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$$

$$=2\pi\int_1^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi\int_1^4 \sqrt{(x)\left(1+\frac{1}{4x}\right)}\,dx$$

$$=2\pi\int_1^4 \sqrt{x+\frac{1}{4}}\,dx$$

$u=x+\frac{1}{4}$ and so on

$$=\cdots=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$